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A NEW METHOD FOR THE SOLUTION OF THE INVERSE PROBLEM OF ELECTRICAL CAPACITANCE TOMOGRAPHY AND ITS APPLICATION TO IMAGE RECONSTRUCTION OF MULTIPHASE FLOWS

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Abstract - A new method is described for solving the inverse problem of parameter identification (2-D) for the generation of images of the distribution of permittivity in the cross section of a pipe for multiphase flow using Electrical Capacitance Tomography. This method needs an *a priori* irrotational condition on the vector of electrical displacement in the flow cross section to obtain the uniqueness of the identification problem from a measurement of the voltage on the surface and using the normal component of the vector of electrical displacement, whose approximate value is obtained from capacitance measurements.
This method takes into account the most important characteristics of the problem, namely: its nonlinearity, its ill posedness, the need for a proper theoretical regularization and the restrictions on the data measurement equipment which induce errors and could prevent one obtaining precise images.
The identification problem is then reduced to solving a linear least square problem with non-convex quadratic restrictions. For this reason, in this new method, it is not necessary to solve the direct problem at each iteration. Finally, a method of low computational cost is obtained which allows the solution of problems in real time for industrial applications, as for example the analysis of the quality of the water-gas-oil mixture during the extraction and refining of petroleum.

1. INTRODUCTION

Electrical Capacitance Tomography (ECT) has recently been tested to obtain image cross-sections of several industrial problems that involve dielectric materials. We are interested in particular in visualizing the cross section of a multiphase flow in pipelines, to be able to determine the permittivity distribution and hence the phase distribution in real time and without any *a-priori* information about the type of flow regime.
The relation between the capacitance C, the permittivity distribution defined on the two-dimensional cross section of the pipe line $\varepsilon(x,y)$ and the generated potential V, can be expressed as

$$C(\varepsilon) = -\frac{1}{V}\int_S \varepsilon(z)\frac{\partial V(z)}{\partial n}ds \qquad (1)$$

where S is the electrode surface and $z=(x,y)$. We can consider this relation as a functional relation between the mutual capacitance measured in the electrode array and the permittivity distribution at the cross section.
Using this relation it is possible to develop alternative methods to solve the inverse problem of determining the permittivity ε using measurements of the capacitance C_{meas}. One possibility is find the unknown permittivity, solving the classical least-squares data fitting minimisation

$$\min f(\varepsilon) = \tfrac{1}{2}\left\|C(\varepsilon) - C_{meas}\right\|^2 \qquad (2)$$

where $C(\varepsilon)$ is the computed capacitance for a given value of ε. However, the solution of problem (2) has the following difficulties:

1) This objective function may be highly non-linear.
2) The number of independent capacitance measurements $N(N-2)/2$, where N is the number of electrodes, is smaller than the needed discretisation of the permittivity distribution, and thus the discretised problem is under-determined and then the solution may not be unique (which is one of the characteristics of the ill-posed problems in the Hadamard sense [9]).
3) Any method used to obtain a particular solution will be unstable due to the fact that the solution is sensitive to measurement errors and noise (which is another characteristic of the ill posedness of this problem [9]).

It is then clear that any method that attempts to solve this problem has to consider the non-linearity and has to use a regularization technique to obtain numerically uniquely stable solutions.

There have been many methods reported in the literature. Some of the methods linearise problem (2) and ignore the non-linear behaviour of the problem. Some other methods, [16], consider the non-linearity and use iterative methods, like Newton-Raphson, iterative Tikhonov, Landweber iteration and others. The problem here is to find the regularization parameter or the sequence of regularization parameters when a linear system is solved at each non-linear iteration. A routine that finds automatically the parameter by constructing an L-curve has been developed [10], but this algorithm has not yet been used in image reconstruction. A unique parameter used in Newton iterative Tikhonov, has to be found very accurately to obtain good images, as will be shown here. Actually this optimal parameter varies strongly from one flow regime to another, and the computation of the regularization parameter becomes the key and the expensive part of image reconstruction and it is still an open question [9].

In this work we describe a completely new method to solve the inverse ECT problem, based on decoupling the original problem, to obtain simpler sub-problems.

This method allow us to recover the permittivity distribution from capacitance measurements, using the only *a-priori* information available that sets bounds on the permittivity and uses the additional condition of non-rotationality of the electrical displacement vector. This condition is observed in pipeline flows [8]. It takes into account the characteristics of the measurement equipment and produces a practical algorithm. In [1]-[3], some of the components of the new method have already been outlined and the purpose of this work is to show the conceptual and numerical advantages of the new method with respect to the traditional least-squares formulation.

The first advantage is the fact that in the new method the solution of the direct problem is not needed at each iteration as it is in the case of the least-squares formulation. The second important advantage is related to the decoupling of the original problem that allows the regularization with known and theoretically founded schemas, which is not the case in the traditional least-squares formulation due to the nonlinearity.

In particular, we will show with numerical examples the high dependency of the spatial distribution of the permittivity obtained with the least-squares formulation to the selection of the Tikhonov regularization parameter. In a future work, we will also show the numerical advantages of the new method.

This paper is organized as follows: in section 2, we describe the model and the inverse parameter identification problem; in section 3 some problems that appear in choosing the regularization parameter are shown when the classical least squares method is used. In section 4, the problem is analysed under the additional condition of non-rotationality of the electrical displacement vector; in section 5 a novel methodology is described which is based on decoupling the original problem into five sub-problems. The conclusions are given in section 6.

2. THE FORWARD AND INVERSE PROBLEM

The design of the cross section of a pipeline for multiphase flow is formed by three concentric circles (see Figure 1), which determine three regions. Within the inner most circle Ω_1 (with radius R_1) we have the multiphase flow with unknown permittivity $\varepsilon_1(x,y)$, whereas in the other annular regions Ω_2 and Ω_3 with exterior radius R_2 and R_3, we have materials with known constant permittivities $\varepsilon_2, \varepsilon_3$. The electrode's array consists of N (8 to 16) contiguous sensing electrodes around the intermediate circle. The generated potential $V^{(i)}$ in this configuration corresponding to the input potential in electrode S_i $i = 1, 2, ..., N,$ has then three components $V_1^{(i)}$, $V_2^{(i)}$ and $V_3^{(i)}$, corresponding naturally to each region.

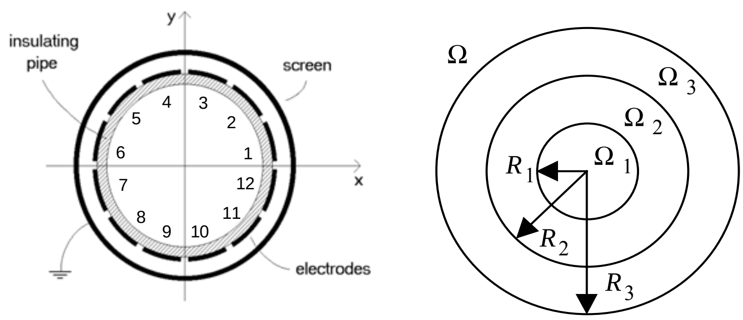

Figure 1. Schematic representation of cross section of pipeline.

The electrodes, being very thin and assumed to have very small gaps between them, are modelled as equipotential surfaces with potential equal to one (lines in the $2-D$ model) covering the entire boundary between Ω_2 and Ω_3. The screen is modelled as an equipotential line on the outer perimeter of Ω_3 with potential equal to zero.

The potential distribution, $\psi^{(i)}$, equal to one at electrode i, is a linear function at the gap between electrode i and its neighbours and it is equal to zero in the rest of $|z| = R_2$ and has the expression (in angular coordinates)

$$\psi^{(i)}(\theta) = \begin{cases} 0 & \text{if} \quad \theta \le \dfrac{2\pi(i-1)}{N} - \dfrac{\theta_0}{2} \\ \dfrac{\theta - \theta_i^-}{\theta_0} + 1 & \text{if} \quad \dfrac{2\pi(i-1)}{N} - \dfrac{\theta_0}{2} \le \theta \le \dfrac{2\pi(i-1)}{N} + \dfrac{\theta_0}{2} \\ 1 & \text{if} \quad \dfrac{2\pi(i-1)}{N} + \dfrac{\theta_0}{2} \le \theta \le \dfrac{2\pi i}{N} - \dfrac{\theta_0}{2} \\ \dfrac{\theta_i^+ - \theta}{\theta_0} + 1 & \text{if} \quad \dfrac{2\pi i}{N} - \dfrac{\theta_0}{2} < \theta \le \dfrac{2\pi i}{N} + \dfrac{\theta_0}{2} \\ 0 & \text{if} \quad \dfrac{2\pi i}{N} + \dfrac{\theta_0}{2} \le \theta \end{cases}$$

where $\theta_i^- = \dfrac{2\pi(i-1)}{N} + \dfrac{\theta_0}{2}, \theta_i^+ = \dfrac{2\pi i}{N} - \dfrac{\theta_0}{2}$ and θ_0 being the angle that separates the electrodes.

The so called Forward Problem consists in given the permittivity distribution $\varepsilon(z)$, find the capacitances $C(\varepsilon)$. This is done solving first the following boundary value problem for the potentials $V_j^{(i)}, \ i = 1,\dots,N$ and $j = 1,,2,3$

$$\nabla.(\varepsilon_j(z)\nabla V_j^{(i)}) = 0 \ \text{ in } \ \Omega_j \quad j = 1,2,3 \tag{3}$$

with the boundary conditions given by

$$V_1^{(i)}(z) = V_2^{(i)}(z) \quad \text{and} \quad \varepsilon_1(z)\frac{\partial V_1^{(i)}}{\partial n_1}(z) = \varepsilon_2 \frac{\partial V_2^{(i)}}{\partial n_1}(z) \tag{4}$$

on $|z| = R_1$, where n_1 is the exterior unit normal vector to the circle.

In addition to eqns (4), we have

$$V_3^{(i)}(z) = V_2^{(i)}(z) = \psi_i \ \text{ in } \ |z| = R_2 \quad \text{and} \quad V_3^{(i)}(z) = 0 \ \text{ in } \ |z| = R_3 \tag{5}$$

Finally, the mutual capacitance values are obtained by the formula

$$C_{i,j}(\varepsilon) = K \int_{S_j} \varepsilon(z) \frac{\partial V^{(i)}}{\partial n_2} ds, \tag{6}$$

where n_2 is the exterior unit unit vector to the circle of radius R_2, K is a constant with units of inverse potential and the electrode area is given by

$$S_i = \left\{ z = (x,y) : |z| = R_2, \ \frac{2\pi(i-1)}{N} + \frac{\theta_0}{2} \le \arg z \le \frac{2\pi i}{N} - \frac{\theta_0}{2} \right\}.$$

We will consider the inverse problem as follows:

Determine the value of $\varepsilon(x, y)$ using model (3)-(6), that reproduce the given $\frac{1}{2}N(N-1)$ measured values of the capacitance C_{meas} $(i, j = 1,2,\dots,N \ \ i < j)$.

3. THE CLASSICAL METHOD OF SOLUTION OF THE INVERSE PROBLEM

In this section we will discuss the difficulties found when solving our inverse problem using the classical least-squares data fitting minimisation to find the unknown permittivity, ε_1, given some measurements of the capacitance C_{ij} in the presence of measurement errors.

This is a non-linear ill-posed inverse problem. Due to the ill-posednes, the measurement errors can be propagated and the optimal solution obtained can be far from the real solution [9]. To avoid this error propagation, it is necessary to regularise the optimisation problem to get the best possible approximation to the permittivity controlling the error propagation. This can be done using for example, the Tikhonov regularisation [9] that depends on a regularization parameter α , modifying the objective function as:

$$\min f(\varepsilon_1) = \tfrac{1}{2}\left\| C(\varepsilon_1) - C_{meas} \right\|^2 + \alpha \left\| L(\varepsilon_1) \right\|^2 \tag{7}$$

where $C(\varepsilon_1)$ is the computed capacitance for a given value of ε_1 , L is a regularisation operator and α is the regularisation parameter.
The solution of this non-linear optimisation problem is calculated iteratively by

$$\varepsilon_1^{k+1} = \varepsilon_1^k + \Delta\varepsilon_1 \tag{8}$$

where the descent direction $\Delta\varepsilon_1$ is found solving the Tikhonov Gauss-Newton linear system of equations and if the initial approximation ε_1^0 is sufficiently close to the solution, the method converges. Conditions to guarantee global convergence from any initial approximation can be found in [12], [13].
The minimisation process will stop when the 2-norm of the gradient satisfies $\left\| \nabla f \right\| < tol_1$ (optimal solution), when no further precision can be achieved $\left\| \varepsilon_1^{k+1} - \varepsilon_1^k \right\| < tol_2$ [14] or, when the Jacobian at some iteration ε_1^k becomes singular (convergences to a non-stationary singular point, [4]).
In order to find the regularisation parameter α in the Tikhonov regularization method, the L-curve algorithm [9],[10] can be used. However, the solution of the inverse problem (7) is very sensitive to the regularisation parameter α, and thus the problem of finding its optimal value accurately is extremely important as will now be shown. Furthermore, as mentioned in the introduction, the permittivity distribution has to be found in real time and considering that in the real world application, the *a-priori* information about the flow regime inside the pipeline may not available, it will be clear with these examples that this traditional least-squares formulation cannot be used to solve the real noisy problem in real time, unless in the future a method to find the regularization parameter for these type of non-linear problems, in a cheaper and theoretically way, is developed.
We will show now, using some results published in [15], that not only the optimal regularization parameter has to be found very precisely (resulting in an expensive procedure) to be able to get an accurate image for the permittivity, but also that this optimal parameter varies by several orders of magnitude from one flow regime to another. Assuming that in practice, there is no information about a possible change of flow regime in the pipeline, it makes it necessary to find the optimal regularization parameter again for each set of measurements, making the real time application almost impossible with this technique.
In order to illustrate the above mentioned difficulties, in [15] the Tikhonov optimal regularisation parameter α was found for several synthetic problems designed for typical two-phase flow regimes, using capacitance data with a random normal distribution error of 2% added. Here, we will only present two of these examples. If the optimal regularization parameter is not found accurately, the images obtained can be completely wrong as will be shown in the first example, that consists of two objects with permittivity $\varepsilon = 2$ placed at the centre (see Figure 2). In the rest of the grid the permittivity is $\varepsilon = 1$. The corner of the L- curve that gives the optimal parameter is shown in Figure 3, and the different images obtained for each value of the Tikhonov parameter are shown in Figure 4, and the optimal parameter of regularization corresponds to $\alpha = 10^{-7}$.
The second example consists of six objects, four at the center and two larger ones at the boundary with permittivity $\varepsilon = 2$. The minimisation process was successful for $\alpha = 10^{-6}$ (see Figures 5-7).
These examples have shown that the order of magnitude of the optimal parameter varies strongly from one flow regime to another, and that only the optimal order of magnitude of the regularization parameter gives the right image. This implies that in a real application where the knowledge of the flow regime is not available, the computation of the optimal regularization parameter cannot be achieved in an automatic way.
Taking these results into consideration, we can conclude that in order to solve our problem (and in many other non-linear cases without the needed *a-priori* information) it is necessary to develop an alternative method to the least-squares formulation in order to transform the problem into a simpler optimisation model that will allow the regularization to be performed in an automatic, efficient and theoretically founded way, without the knowledge of the flow regime. We will now present the needed background to proceed to present the new method.

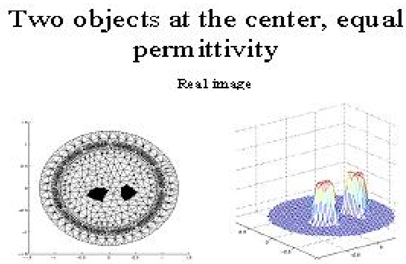

Fig. 2 Real object

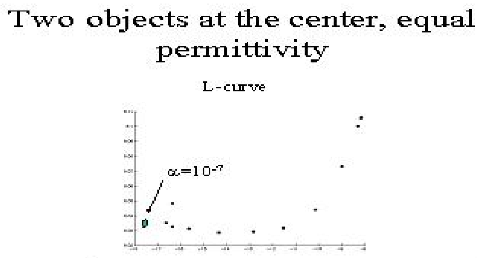

Fig. 3 L-curve to find the optimal α

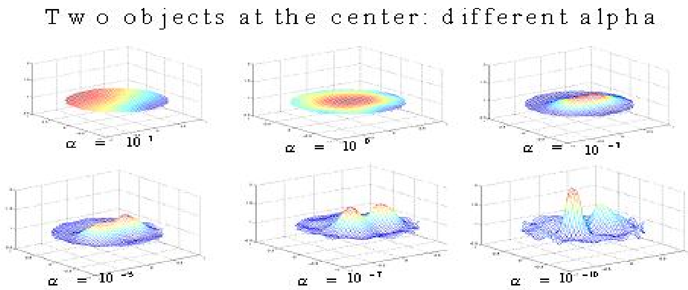

Fig. 4 The calculated objects for different α

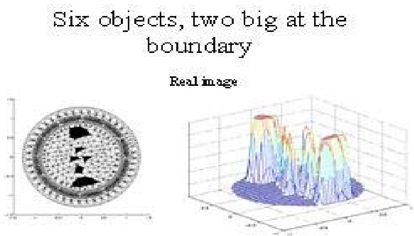

Fig. 5 Real object

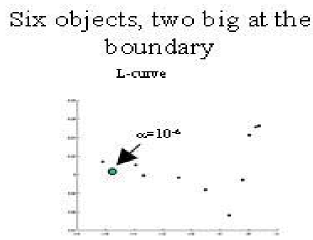

Fig. 6 L-curve to find the optimal α

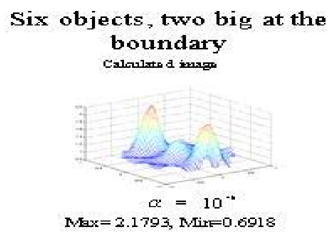

Fig. 7 The calculated objects for the optimal α

4. ADDITIONAL ASSUMPTIONS AND RESULTS

To propose a novel method, we need the additional assumption that the field of electrical displacement $J^{(i)} = \varepsilon_1 \nabla V_1^{(i)}$ satisfies that $rot\ J^{(i)} = 0$, that is, $J^{(i)}$ is non-rotational, which can be considered an acceptable practical criterion in our application if the flow regime is not highly complex [8]. The non-rotationality of $J^{(i)}$, together with (3) for $j = 1$, imply the existence of a harmonic function u_1 in Ω_1 such that

$J^{(1)} = \nabla u_1$. From (4), this harmonic function u_1 satisfies the Neumann boundary condition $\dfrac{\partial u_i}{\partial n_1} = \varepsilon_2 \dfrac{\partial V_2^{(i)}}{\partial n_1}$ at the boundary of the circle of radius R_1, and u_1 is given by

$$u_i(z) = -\frac{\varepsilon_2}{2\pi} \int_{|\xi|=R_1} \frac{\partial V_2^{(i)}}{\partial n_1}(\xi) \ln|z-\xi| ds_\xi \,. \tag{10}$$

Applying the gradient operator to each side of this equation and taking the modulus we obtain the expression for ε_1

$$\varepsilon_1(z) = \frac{\dfrac{\varepsilon_2}{\pi}\left| \displaystyle\int_{|\xi|=R_1} \frac{\partial V_2^{(i)}}{\partial n_1}(\xi) \nabla_z \left(\ln|z-\xi|\right) ds_\zeta \right|}{\left|\nabla V_1^{(i)}\right|} \tag{11}$$

Then, if we know $V_1^{(i)}$ and $V_2^{(i)}$ we can obtain ε_1 from (11). In the following we describe how to obtain $V_1^{(i)}$ in this method and thus, to obtain a tomographic image of a cross section on a pipeline.

In order to transform the boundary value problem (3)-(5) into an equivalent system of integral equations that will allow us to build a functional to find $V_1^{(i)}$ from the measured capacitances, we will use the boundary conditions (4) and Green functions.

Let $N(z,\xi)$ be the Green function in Ω_2 that satisfies $N(z,\xi) = \delta(z-\xi)$ in $z,\xi \in \Omega_1$ and the boundary conditions $\dfrac{\partial N}{\partial n_z}(z,\xi) = 0$ in $|z| = R_1$, $\xi \in \Omega_2$ and $N(z,\xi) = 0$ in $|z| = R_2$, $\xi \in \Omega_2$, where $\delta(z-\xi)$ is Dirac's delta function. Applying the Green formula to the Laplace operator for N and $V_2^{(i)}(\xi)$ in Ω_2, we obtain the expression

$$V_2^{(i)}(z) = \int_{|\xi|=R_1} N(z,\xi) \frac{\partial V_2^{(i)}}{\partial n_1}(\xi) ds_\xi + \int_{|\xi|=R_2} \frac{\partial N}{\partial n_2}(z,\xi)\, \psi_i(\xi) ds_\xi \tag{12}$$

Applying the Green formula to the pairs $\left(V_1^{(i)}, d\right)$ and $\left(V_1^{(i)}, n\right)$, where $d(z,\xi)$ and $n(z,\xi)$ are the Green functions corresponding to the Dirichlet and Neumann problems for the Laplace operator in the disk Ω_1 and using the expressions (10) and (12), the following relations are obtained

$$\begin{aligned} -V_1^{(i)}(z) = &\int_{|\xi|<R_1} \nabla V_1^{(i)}(\xi) \cdot \nabla_\xi d(z,\xi)\, d\xi + \frac{\varepsilon_2}{\pi} \int_{|\varphi|<R_1} \left[\int_{|\xi|<R_1} \nabla_\xi \ln|\xi-\varphi| \nabla_\xi d(z,\varphi) d\xi \right] \frac{\partial V_2^{(i)}}{\partial n_1}(\varphi) ds_\varphi \\ &- \int_{|\xi|=R_1} V_2^{(i)}(\xi) \frac{\partial d}{\partial n_1}(z,\xi) ds_\xi \end{aligned} \tag{13}$$

$$\begin{aligned} -V_1^{(i)}(z) = &\int_{|\xi|<R_1} \nabla V_1^{(i)}(\xi) \cdot \nabla_\xi n(z,\xi)\, d\xi + \frac{\varepsilon_2}{\pi} \int_{\lfloor\varphi\rfloor<R_1} \left[\int_{|\xi|<R_1} \nabla_\xi \ln|\xi-\varphi| \nabla_\xi n(z,\varphi) d\xi \right] \frac{\partial V_2^{(i)}}{\partial n_1}(\varphi) ds_\varphi \\ &- \frac{\varepsilon_2}{2\pi R_1} \int_{|\xi|=R_1} V_2^{(i)}(\xi) ds_\xi \end{aligned} \tag{14}$$

The following result allows us to use the system of integral eqns (10), (12)-(14) to build a functional to find $V_1^{(i)}$ from the measured capacitances.

Theorem 1. For each strictly positive function ε_1 of class $C^2(\Omega_1)$ and continuous in $|z| \le R_1$, the unique classical solution of the boundary value problem (3)-(5) for $V_1^{(i)}(z)$ and $V_2^{(i)}(z)$, for $i = 1,...,N$, coincides with the solution of the integral equations system (10), (12)-(14) if the additional condition of non-rotationality of $J^{(i)} = \varepsilon_1 \nabla V_1^{(i)}$ is satisfied. The proof of this theorem has been given in [2]

5. DESCRIPTION OF THE METHOD

In this section we will describe the several steps that compounds the novel method and the details will be studied in a later paper. The novel method computes first $V_3^{(i)}$, then $V_2^{(i)}$ to finally get $V_1^{(i)}$ which allows computing the desired permittivity distribution ε_1 by the secondary relation (11). The method consists in the following steps:

a) Obtain $V_3^{(i)}$ in Ω_3 solving the Dirichlet Boundary Value problem (3) (for $j = 3$) with (5) in the annulus $R_2 < |z| < R_3$. The solution to this problem is obtained using a Fourier Series that uniformly converges due to the boundary condition ψ_i.

b) Obtain $\frac{\partial V_2^{(i)}}{\partial n_2}$ using a stable interpolation algorithm like the local spline-approximation method [5] from the known integral data $v_{i,j}$

$$v_{i,j} = \int_{S_j} \frac{\partial V_2^{(i)}}{\partial n_2} ds = \frac{1}{\varepsilon_2} \left\{ \varepsilon_3 \int_{S_j} \frac{\partial V_3^{(i)}}{\partial n_2} ds - \frac{C_{i,j}}{K} \right\}, \tag{15}$$

obtained from (6) . The solution for $V_3^{(i)}$ would allow us to substitute in (15) the value of $V_3^{(i)}$ by a partial sum. This is computed only once at the beginning.

c) Using the computed value of $\frac{\partial V_2^{(i)}}{\partial n_2}$ and the known function ψ_i over the circle of radius R_2 (input potential), a Cauchy problem is solved for the Laplace equation in the annular region Ω_2 to obtain $V_2^{(i)}$. There exist stable procedures via regularization methods, to obtain approximate values of $V_2^{(i)}$ and its normal derivative $\frac{\partial V_2^{(i)}}{\partial n_1}$ on the circle of radius R_1, see [6], [9], [11].

d) The discretisation grid for Ω_1 can be selected using information on the measurement errors given by the equipment and on the number of electrodes used, as well as the required accuracy of the solution. In this form we obtain a number M of grid elements. The number M is equal to the maximum number of elements of a partition $\{\omega_k\}_{k=1}^{M}$ of the region Ω_1 in which each component ω_k is such that the instrument used to measure the mutual capacitances can detect the change of ε_1^k in ω_k, where ε_1^k is the value of $\varepsilon_1(z)$ assigned to the component ω_k as a result of the discretisation.

e) The boundary value problem (3)-(4) is reduced to the system of integral eqns (10), (12)-(14) and $V_1^{(i)}$ is computed by introducing a functional that takes into account the relation (10), after applying the gradient operator to (10) and substituting the value of $V_2^{(i)}$ in $|z| = R_1$ by the approximate expression (12). After performing these substitutions, we obtain the relations:

$$-V_1^{(i)}(z) = \int_{|\xi|<R_1} \nabla V_1^{(i)}(\xi) \cdot \nabla_\xi d(z,\xi)\, d\xi + \int_{|\varphi|=R_1} F(z,\varphi) \frac{\partial V_2^{(i)}}{\partial n_1}(\varphi) ds_\varphi + h_i(z) \tag{16}$$

$$-V_1^{(i)}(z) = \int_{|\xi|<R_1} \nabla V_1^{(i)}(\xi) \cdot \nabla_\xi n(z,\xi)\, d\xi + \int_{|\varphi|=R_1} G(z,\varphi) \frac{\partial V_2^{(i)}}{\partial n_1}(\varphi) ds_\varphi + a_i \tag{17}$$

where

$$F(z,\varphi)=\frac{\varepsilon_2}{\pi}\int_{|\xi|<R_1}\nabla_\xi \ln|\xi-\varphi|\cdot\nabla_\xi d(z,\xi)d\xi-\int_{|\psi|=R_1}N(\psi,\varphi)\frac{\partial d}{\partial n_\psi}(z,\psi)ds_\psi, \tag{18}$$

$$G(z,\varphi)=\frac{\varepsilon_2}{\pi}\int_{|\xi|<R_1}\nabla_\xi \ln|\xi-z|\cdot\nabla_\xi n(z,\xi)d\xi+\varepsilon_2 n(z,\varphi)-\frac{1}{2\pi R_1}\int_{|\psi|=R_1}N(\psi,\varphi)ds_\psi \tag{19}$$

$$h_i(z)=-\int_{S_i}\left\{\int_{|\psi|=R_1}\frac{\partial d}{\partial n_\psi}(z,\psi)\frac{\partial N}{\partial n_{\xi_1}}(\psi,\xi_1)ds_\psi\right\}ds_{\xi_1} \tag{20}$$

and

$$a_i=-\frac{1}{2\pi R_1}\int_{S_i}\left\{\int_{|\psi|=R_1}\frac{\partial N}{\partial n_{\xi_1}}(\psi,\xi_1)ds_\psi\right\}ds_{\xi_1} \tag{21}$$

From relations (10), (16) and (17) a functional is introduced such that, assuming that $V_2^{(i)}$ and $\frac{\partial V_2^{(i)}}{\partial n_1}$ have been computed exactly on $|z|=R_1$, it reaches its minimum on the solution $V_1^{(i)}$ of problem (3)-(5) for $i=1,2.$
In fact, from (10) we obtain the relation:

$$f_i(z)\frac{\partial V_1^{(i)}}{\partial y}(z)=g_i(z)\frac{\partial V_1^{(i)}}{\partial x}(z) \tag{22}$$

where, if $\xi=R_1e^{i\tau}$

$$f_i(z)=\int_{|\xi|=R_1}\frac{\partial V_2^{(i)}}{\partial n_1}(\xi)\frac{x-R_1\cos\tau}{|\xi-z|^2}ds_\xi \tag{23}$$

$$g_i(z)=\int_{|\xi|=R_1}\frac{\partial V_2^{(i)}}{\partial n_1}(\xi)\frac{y-R_1\sin\tau}{|\xi-z|^2}ds_\xi \tag{24}$$

Then a functional is defined, for each $i=1,...N$, through

$$\begin{aligned}l(V_1^{(i)})=&\left\|f_i(z)\frac{\partial V_1^{(i)}}{\partial y}(z)-g_i(z)\frac{\partial V_1^{(i)}}{\partial x}(z)\right\|^2+\left\|V_1^{(i)}(z)+\int_{|\xi|<R_1}\nabla_\xi V_1^{(i)}(\xi)\cdot\nabla_\xi d(z,\xi)d\xi+F_i(z)\right\|^2\\&+\left\|V_1^{(i)}(z)+\int_{|\xi|<R_1}\nabla_\xi V_1^{(i)}(\xi)\cdot\nabla_\xi n(z,\xi)d\xi+G_i(z)\right\|^2\end{aligned} \tag{25}$$

where $\|\cdot\|$ represents the norm in $L_2(\Omega_1)$ and

$$F_i(z)=\int_{|\varphi|=R_1}F(z,\varphi)\frac{\partial V_2^{(i)}}{\partial n_1}(\varphi)ds_\varphi+h_i(z) \tag{26}$$

$$G_i(z)=\int_{|\varphi|=R_1}G(z,\varphi)\frac{\partial V_2^{(i)}}{\partial n_1}(\varphi)ds_\varphi+a_i \tag{27}$$

Functional (25) is minimized subject to the bounded constraints:

$$0<\varepsilon_{\min}\le\varepsilon_1^{(i)}(z)\le\varepsilon_{\max} \tag{28}$$

where $\varepsilon_{\min}$ and $\varepsilon_{\max}$ are positive numbers fixed *a-priori*, depending on the permittivity values for the multiphase mix components, that we need to recover.

f) The next step consists in the discretisation of functional (25) and the constraints (28) after the application of a convenient collocation method. We search $V_1^{(i)}$ in the approximate form:

$$V_1^{(i)}(z) = \sum_{k=1}^{M} a_k^{(i)} W_k(z) \tag{29}$$

where $W_k(z)$ are the eigen-functions corresponding to the Neumann problem for the Laplace operator in the circle Ω_1 and the coefficients $a_k^{(i)}$ are unknown. Substituting (29) in the expression of functional (25) we get a new functional that is a finite dimensional approximation of this functional

$$L\left(a^{(i)}\right) = \left|F_i a^{(i)}\right|_M^2 + \left|Da^{(i)} + b^{(i)}\right|_M^2 + \left|Na^{(i)} + c^{(i)}\right|_M^2 \tag{30}$$

where $|\cdot|_M$ denotes the vector modulus in $\Re^M$ and $F_i = \left(f_{jk}^{(i)}\right)$, $D = (D_{jk})$, $N = (N_{jk})$, $a^{(i)} = (a_1^{(i)},...,a_M^{(i)})^t$, $b^{(i)} = (F_1^{(i)},...,F_M^{(i)})^t$ and $c^{(i)} = (G_1^{(i)}, G_2^{(i)},...,G_M^{(i)})^t$, $f_{jk}^{(i)}, D_{jk}, N_{jk}$, $F_j^{(i)}$ and $G_j^{(i)}$ are the Fourier coefficients of the functions $f_k^{(i)}(z), D_k(z), N_k(z)$, $F_i(z)$, $G_i(z)$ with respect to the system $\{W_j\}$, $F_i(z)$ and $G_i(z)$ given by (26) and (27) and

$$f_k^{(i)}(z) = f_i(z)\frac{\partial W_k(z)}{\partial y} - g_i(z)\frac{\partial W_k(z)}{\partial x}, \qquad D_k(z) = W_k(z) + \int_{|\zeta|<R_1} \nabla_\zeta W_k(\zeta)\cdot\nabla_\zeta d(z,\zeta)d\zeta$$

$$N_k(z) = W_k(z) + \int_{|\zeta|<R_1} \nabla_\zeta W_k(\zeta)\cdot\nabla_\zeta n(z,\zeta)d\zeta$$

The choice of the basis $\{W_k(z)\}_{k=1}^{\infty}$ reduces the degree of ill-posedness of the problem, since the differentiation of the potential , necessary for identification of permittivity ε_1 by (11), regarding this basis, is a stable procedure. Now, from relation (10), we obtain that constraints (32) are equivalent to the inequalities:

$$\frac{\varepsilon_2 R_1}{\sqrt{2\pi\varepsilon_{\max}}} \le \left|A_i a^{(i)}\right| \le \frac{\varepsilon_2 R_1}{\sqrt{2\pi\varepsilon_{\min}}} \tag{31}$$

$$\frac{\varepsilon_2 R_1}{\sqrt{2\pi\varepsilon_{\max}}} \le \left|B_i a^{(i)}\right| \le \frac{\varepsilon_2 R_1}{\sqrt{2\pi\varepsilon_{\min}}} \tag{32}$$

where $A_i = (\alpha_{jk}^{(i)})$, $B_i = (\beta_{jk}^{(i)})$ and $\alpha_{jk}^{(i)}$, $\beta_{jk}^{(i)}$ are the Fourier coefficients of $\frac{1}{f_i(z)}\frac{\partial W_R}{\partial x}$ and $\frac{1}{g_i(z)}\frac{\partial W_R}{\partial x}$ respectively, with respect to $W_j(z)$. Finally the optimisation problem that has to be solved to obtain the coefficients $a_k^{(i)}$ of the approximate expression (29) of $V_1^{(i)}(z)$, is to minimize the functional (30) subject to the constraints (31) and (32).

g) Once $V_1^{(i)}$ has been obtained, the computation of $\varepsilon_1^{(i)}$ can be made numerically using (11). Function $\varepsilon_1^{(i)}$ has to be evaluated at each element of the grid, which is used to obtain a tomographic image of the cross section of the pipeline.

6. CONCLUSIONS

1. In section 4 of this work it has been shown that the classical method of solution presents some difficulties to choose the optimal regularization parameter due to ill-posedness and non-linearity of the problem and the fact that there is no *a priori* information about the type of flow.

2. The proposed method takes into account the important characteristics of the problem: its non-linearity, its ill-posedness due to the intrinsic properties of the identification problem and the lack of data, the need to use a theoretically justified regularization method. These difficulties have not yet been solved completely by any of the existing methods as is discussed in the introduction. We see that the non-rotationality condition, which is satisfied for many dielectrical multi-phase flows in pipes, allows the decomposition of the inverse problem into

several simpler problems, and each one of them can be solved with a numerically stable procedure, with the necessary theoretical justification. More importantly for these simpler problems, we use a projective method to express the unknown potential with respect to a smooth basis of functions, that reduce the degree of ill-posedness of the problem, since the differentiation of the potential, necessary for identification of permittivity, regarding this basis, is a stable procedure.

3. Here it is not necessary to solve the direct problem at each iteration, as in the classical method of solution of the inverse problem. The optimisation problem in this method is minimizing a linear least squares functional subject to quadratic constraints and, as a consequence, the low computational cost will then allow the solution of this problem in real time for industrial applications.

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